

UNIVERSAL ARROWS TO FORGETFUL FUNCTORS FROM CATEGORIES OF TOPOLOGICAL ALGEBRA

VLADIMIR G. PESTOV

Department of Mathematics
Victoria University of Wellington
P.O. Box 600
Wellington, New Zealand

vladimir.pestov@vuw.ac.nz

ABSTRACT. We survey the present trends in theory of universal arrows to forgetful functors from various categories of topological algebra and functional analysis to categories of topology and topological algebra. Among them are free topological groups, free locally convex spaces, free Banach-Lie algebras, and more. An accent is put on relationship of those constructions with other areas of mathematics and their possible applications. A number of open problems is discussed; some of them belong to universal arrow theory, and other may become amenable to the methods of this theory.

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INTRODUCTION

The concept of a universal arrow was invented by P. Samuel in 1948 [Sa] and put in connection with his investigations on free topological groups. The following definition is taken from the book [MaL].

Definition. If $S : D \rightarrow C$ is a functor and c an object of C , a universal arrow from c to S is a pair $\langle r, u \rangle$ consisting of an object r of D and an arrow $u : c \rightarrow Sr$ of C , such that to every pair $\langle d, f \rangle$ with d an object of D and $f : c \rightarrow Sd$ an arrow of C , there is a unique arrow $f' : r \rightarrow d$ of D with $Sf' \circ u = f$.

In other words, every arrow f to S factors uniquely through the universal arrow u , as in the commutative diagram

$$\begin{array}{ccc} c & \xrightarrow{u} & Sr \\ \parallel & & \downarrow Sf' \\ c & \xrightarrow{f} & Sd \end{array}$$

□

This notion bears enormous generality and strength, and at present it is an essential ingredient of category theory [MaL] and theory of toposes [Joh]. In fact, many mathematical constructions can be interpreted as universal arrows of one of another kind. Examples are: quotient structures and substructures, products and coproducts, including algebraic and topological tensor products, universal enveloping algebras, transition from a Lie algebra to a simply connected Lie group and *vice versa*, compactifications of all kinds (Stone-Čech, Bohr, and others), completions, prime spectra of rings, and much more.

We are interested in the particular case where S is a *forgetful functor* from some category of topological algebra or functional analysis, D , to another category of topological algebra or functional analysis or a category of topology, C . Historically the first, and studied in most detail, is the construction of the *free topological group*, $F(X)$, over a topological space X , where C is the category of Tychonoff topological spaces and continuous mappings and D is the category of Hausdorff topological groups and continuous homomorphisms. A number of similar constructions have received a comprehensive treatment, among them are free Abelian topological groups, free compact groups, free locally convex spaces. At the same time, in recent years similar constructions have arisen — either explicitly or implicitly — in other areas of mathematics. In some cases no attempt has been made to establish a bridge between those and former types of universal arrows — although seemingly such a connection would facilitate a study of new constructions. Among the disciplines where new types of universal arrows to forgetful functors are likely to play a noticeable role, are infinite-dimensional Lie theory, supermanifold theory, differential geometry, C^* -algebras and “quantized” functional analysis.

We do not aim at a comprehensive presentation of the subject outlined in the title of this paper, nor we give detailed proofs of the results: such an elaborate approach would lead to a voluminous treatise. Instead, we discuss a few carefully selected lines of development which, as we see it, dominated the research over more than 50 years. We are focussing on the most interesting unsolved problems. Also, we do our best in forecasting the future directions of the theory, paying special attention

to recent germs of it in areas of mathematics bordering topological algebra (Lie theory, functional analysis and mathematical physics).

This small survey inevitably tends to the results and ideas coming from the Russian (or, in a more politically correct language, ex-Socialist, to cover Ukrainian, Moldavian, Georgian, Bulgarian and other contributors) school of universal arrow theorists, where the author himself comes from. Most probably and to the author's regret, the contributions from the other two major centers — the Australian and the American schools — were underrepresented in this article. As a matter of fact, the author's personal tastes and research work of his own were prevalent in selecting topics for discussion.

Our bibliography, although (intentionally) not complete, is hopefully “everywhere dense” in the subject (a comparison due to Kac [Kac1]).

1. MAJOR CLASSICAL EXAMPLES

The following are major examples of universal arrows to forgetful functors from categories of topological algebra, which are subject of a traditional study in this area. We are marking with a lozenge (\diamond) those notions which will be later considered in this survey to some extent. By abuse of terminology and notation, we will sometimes identify a universal arrow with its target object (no confusion should however result from that).

1. \diamond $C = \mathbf{Tych}$ (the category of Tychonoff topological spaces and continuous mappings) and $D = \mathbf{TopGrp}$ (the category of Hausdorff topological groups and continuous homomorphisms). The universal arrow from an object $X \in C$ (a Tychonoff space) to the forgetful functor $S : D \rightarrow C$ is the *(Markov) free topological group over X , $F(X)$* .

This notion was introduced in 1941 by Markov [Mar1] who presented his results in most detail somewhat later [Mar2]. Among those mathematicians who responded first to the new concept, were Nakayama [Nak], Kakutani [Kak], Samuel [Sa] and Graev [Gr1]; the latter work has had an enormous impact on later investigations in the area, and the paper by Samuel, as we have already mentioned, has produced a deep methodological insight.

2. $C = \mathbf{Tych}_*$ (the category of pointed Tychonoff topological spaces and continuous mappings preserving base points) and $D = \mathbf{TopGrp}$ (the base point of a topological group being e , the identity). The universal arrow from an object $X \in C$ (a pointed Tychonoff space) to the forgetful functor $S : D \rightarrow C$ is the *Graev free topological group over X , $F_G(X)$* .

In fact, the Markov and Graev free topological groups are very closely related to each other by means of the following short exact sequence:

$$e \rightarrow \mathbb{Z} \rightarrow F_M(X) \rightarrow F_G(X) \rightarrow e$$

The choice of a basepoint $* \in X$ does not affect the topological group $F_G(X)$. The Markov free group of X is isomorphic to the Graev free group of the disjoint sum $X \oplus \{*\}$. [Gr1,2]. This is why we consider the Markov free topological groups only. Anyway, the Graev approach seems more convenient in some other cases such as free Banach spaces and free Banach-Lie algebras over metric spaces.

3. $C = \mathbf{Met}_*$ (the category of pointed metric spaces) and $D = \mathbf{MetGrp}$ (the category of groups endowed with bi-invariant metric). The universal arrow from

an object $(X, \rho, *) \in C$ (a pointed metric space) to the forgetful functor $S : D \rightarrow C$ is the *free group over $X \setminus \{*\}$ endowed with the Graev metric $\bar{\rho}$* .

This concept is due to Graev [Gr1,2]. The metrized group $(F(X), \bar{\rho})$ is of no particular interest by itself; it deserves attention as an auxiliary device for studying the free topological group $F(X)$. An amazing example of such kind is the Arhangel'skii's theorem from [Arh4]. If one wants to consider Graev metrics on a Markov free group then one should start with a fixed metric ρ on the set $X \oplus \{e\}$.

4. $C = \mathbf{Tych}$ and $D = \mathcal{V}$ is a *variety* of Hausdorff topological groups, considered as a subcategory of \mathbf{TopGrp} . The universal arrow from an object $X \in C$ (a Tychonoff space) to the forgetful functor $S : D \rightarrow C$ is the *free topological group over X in the variety \mathcal{V} , $F_{\mathcal{V}}(X)$* .

Varieties of topological groups can be understood in different sense (cf. [Mo1,2,10] and [Pr2,3,PrS]). It would not be clear what is the “right” notion until a non-disputable version of the Birkhoff theorem for topological groups is obtained (see, however, [Ta]). Anyway, all of the most important classes of topological groups fit both definitions. Examples of varieties are: the variety of SIN groups (topological groups with equivalent left and right uniformities) [MoTh1]; that of topological groups with quasi-invariant basis [Kats1] ($= \aleph_0$ -balanced groups in [Arh5]); of totally bounded, or precompact, groups; of \aleph_0 -bounded groups [Gu, Arh5] etc. There is a survey on free topological groups in varieties [Mo10]. A free topological group, $F_{\mathcal{V}}(X)$, in a variety \mathcal{V} is actually the composition of the universal arrow $F(X)$ and the universal arrow from $F(X)$ to the natural embedding functor $\mathcal{V} \rightarrow \mathbf{TopGrp}$. The notion of a free topological group relative to classes of topological groups, considered by Comfort and van Mill [ComvM], can be redefined in terms of free topological groups in relevant varieties, and the questions of existence of such free topological groups completely reduces to certain questions about free topological groups in varieties.

The following is the most important particular case.

5. $C = \mathbf{Tych}$ and $D = \mathbf{AbTopGrp}$ (the category of Abelian topological groups and continuous homomorphisms). The universal arrow from an object $X \in C$ (a Tychonoff space) to the forgetful functor $S : D \rightarrow C$ is the *(Markov) free Abelian topological group over X , $A(X)$* .
6. $C = \mathbf{Tych}_*$ and $D = \mathbf{AbTopGrp}$. The universal arrow from an object $X \in C$ (a Tychonoff space) to the forgetful functor $S : D \rightarrow C$ is the *Graev free Abelian topological group over X , $A_G(X)$* .

Of course, $A(X)$ (resp. $A_G(X)$) is just the abelianization of $F(X)$ (resp., $F_G(X)$).

7. $C = \mathbf{Tych}$ and $D = \mathbf{CompGrp}$ (the category of compact topological groups and continuous homomorphisms). The universal arrow from an object $X \in C$ (a Tychonoff space) to the forgetful functor $S : D \rightarrow C$ is the *free compact group over X , $F_C(X)$* .

Remark that the free compact group $F_C(X)$ is nothing but the Bohr compactification, $bF(X)$, of the free topological group, $F(X)$. (The Bohr compactification, bG , of a topological group G [Mo9] is the universal arrow from G to the embedding functor $\mathbf{CompGrp} \rightarrow \mathbf{TopGrp}$.) We do not touch free compact groups in our survey, and refer the reader to the series of papers by Hofmann and Morris [HfHM01,5]. Also free compact groups may be viewed as completions of free

precompact groups (or, just the same, free totally bounded groups), that is, free topological groups in the correspondent variety. Free precompact groups have been studied recently in connection with some questions of dimension theory [Sh].

Of course, the notion of the free compact Abelian group over X also makes sense, and the structure of such groups has been described in detail (*loc. cit.*).

8. \diamond $C = \mathbf{Unif}$ is the category of uniform spaces and $D = \mathbf{TopGrp}$. There are at least four “natural” forgetful functors from D to C ; our choice as S is the functor S assigning to a topological group G the *two-sided* uniform structure on it; we will denote the resulting uniform space by G_t . The universal arrow from an object $X \in \mathbf{Unif}$ (a uniform space) to the functor S is the *free topological group over X* , or the *uniform free topological group*, $F(X)$.

This was an invention of Nakayama [Nak]. Free topological groups over uniform spaces later proved to be a most natural framework for analysing some aspects of free topological groups, see [Nu2]. Free topological groups over uniform spaces provide a straightforward generalization of free topological groups over Tychonoff spaces, because for a Tychonoff space X the free topological group over X is canonically isomorphic to the free topological group over the finest uniform space associated to X .

9. By replacing the category \mathbf{Tych} by \mathbf{Unif} in the items 2,4,6 one comes to the obviously defined concepts of a (*Graev*) *free (Abelian) topological group over a uniform space*.
10. $C = \mathbf{Tych}$ (resp., \mathbf{Unif}) and $D = \mathbf{LCS}$ (the category of locally convex spaces and continuous linear operators). The universal arrow from an object $X \in C$ (a Tychonoff space; resp., a uniform space) to the forgetful functor $S : D \rightarrow C$ (which in the second case is also defined unambiguously, unlike in item 8) is the *free locally convex space over a topological (uniform) space X* , and is denoted by $L(X)$.

This concept is also an invention of Markov [Mar1]. However, for some reason it received no immediate attention from mathematical community until the paper by Raïkov [Rai2]. The most important of later developments is due to Uspenskiĭ [U2]. A particular case of this construction — the notion of a vector space endowed with finest locally convex topology — is well known in functional analysis [Sch]; it is actually the free locally convex space over a discrete topological space X .

11. As in item 4, one can consider universal arrows from an object of \mathbf{Tych} to the forgetful functor $\mathcal{V} \rightarrow \mathbf{Tych}$ where \mathcal{V} is a *variety* of locally convex spaces in one or another sense. We denote the resulting *free locally convex space over X in the variety \mathcal{V}* by $L_{\mathcal{V}}(X)$.

We refer the reader to a very solid paper [DmOS] by Diestel, Morris and Saxon, and a survey [Mo10] by Morris. Other references include [Ber].

12. \diamond If \mathcal{V} is the variety of locally convex spaces with weak topology then the resulting *free locally convex space with weak topology* over a Tychonoff space X is denoted by $L_p(X)$.

This concept seemingly was well known in functional analysis for decades, because the space $L_p(X)$ is the weak dual of the space $C_p(X)$ of continuous functions on X in the topology of pointwise (simple) convergence. See, e.g., [Wh] and references therein.

13. $C = \mathbf{Met}_*$ and $D = \mathbf{Ban}$ is the category of *complete normed linear spaces and linear operators of norm ≤ 1* . The universal arrow from an object $X \in \mathbf{Met}_*$ to the forgetful functor $S : D \rightarrow C$ (the origin is a base point) is the *free Banach space over a pointed metric space*, $B(X)$.

This object first appeared in the paper by Arens and Eells [ArE]; see also [Rai2; Pe9]. However, it was considered by functional analysts independently and at a different angle of view: the normed space $B(X)$ is known as the *predual of the space $Lip(X)$ of Lipschitz functions* on a pointed metric space X .

14. $C = \mathbf{Tych}$ and D is the category of universal topological algebras of a given signature Ω . In this case the universal arrow from a space X to the forgetful functor $D \rightarrow C$ is the *free universal topological algebra* over X .

Such algebras were first considered by Mal'cev [Mal] and others [Ta, Pr2, PrS]. We will not touch them in our survey.

15. $C = \mathbf{Tych}$ and D is the category of topological associative rings or associative algebras. The resulting *free topological rings* and *free topological algebras* have been also considered by Arnautov, Mikhalev, Ursul and others [AMV].

Later in our survey we will consider also a number of less traditional examples of universal arrows to forgetful functors. All of them are universal arrows to forgetful functors of one or another kind. The following notion, that of free product of topological groups, at first seems not to fit into this scheme.

16. \diamond Let $C = \mathbf{TopGrp} \times \mathbf{TopGrp}$, $D = \mathbf{TopGrp}$, and let S be the diagonal functor $\mathbf{TopGrp} \rightarrow \mathbf{TopGrp} \times \mathbf{TopGrp}$. (That is, $S(G) = (G, G)$.) The universal arrow from a pair (G, H) of topological groups to the functor S is called the *free product of G and H* and denoted by $G * H$. In other terms, $G * H$ is just the coproduct of G and H in the category \mathbf{TopGrp} .

Anyway, it is well known that this notion (belonging to Graev [Gr3]) is of the same nature as that of a free topological group, those constructions share a number of common properties and indeed, it can be (if necessary) reshaped as a universal arrow to an appropriate *forgetful* functor. Let $C = \mathbf{TopGrp} \times \mathbf{TopGrp}$ be as above, and let D denote the category of all topological groups with two fixed subgroups. Then $G * H$ can be viewed as the universal arrow from a pair (G, H) to the forgetful functor from D to C which forgets the first group and sends a triple (F, G, H) to (G, H) .

17. \diamond In an obvious way, the concept of the free product can be defined for arbitrary families of topological groups, $\{G_\alpha : \alpha \in A\}$. This product is denoted by $*_{\alpha \in A} G_\alpha$.

In all the aforementioned cases, similar methods, which are actually of a categorical nature, are used to prove the existence, uniqueness and a number of other properties of universal arrows. We will summon those results as follows.

1.1. Theorem.

- (1) In all cases 1–17 the universal arrow exists and is unique.
- (2) In all cases apart from 4, the universal arrow is an isomorphic embedding.
- (3) In case 4, the universal arrow is a homeomorphic embedding if the variety \mathcal{V} contains at least one non totally path-disconnected topological group.
- (4) In all cases apart from 4 and 7, the image of the universal arrow is topologically closed. \square

2. STRUCTURE OF FREE TOPOLOGICAL GROUPS

Among the first, and most vital, questions to be asked about any universal arrow to forgetful functor from a category of topological algebra, is the question of description of the algebro-topological structure of the target object of this arrow. In some cases such a description poses no serious problems, but for most (especially noncommutative) examples it is rather challenging. Since this question seems to be best investigated for free topological groups, we find it necessary — and very instructive — to survey the state of affairs in this area.

1. Description of topology. The topology of a free topological group $F(X)$ is rather complicated, and among the achievements of Graev [Gr1,2] was a description of the topology of $F(X)$ in the case where X is a compact space. Later his description was transferred to the so-called k_ω -spaces by Mack, Morris and Ordman [MaMoO], which result has substantially widened the sphere of applicability of the original description. We will give it in the strongest form.

Denote by $\tilde{X} = X \oplus -X \oplus \{e\}$ the disjoint sum of a Tychonoff space X , of its topological copy $-X = \{-x : x \in X\}$, and a one-point space $\{e\}$. For each $n = 0, 1, 2, \dots$ there is an obviously defined canonical continuous mapping $i_n : \tilde{X}^n \rightarrow F(X)$. Denote by $F_n(X)$ the subspace of $F(X)$ image of i_n ; it is closed. A topological space X is called a k_ω -space if it can be represented as a union of countably many compact subsets X_n in such a way that the topology of X is a *weak* topology with respect to the cover $\{X_n : n \in \mathbb{N}\}$, that is, a subset $A \subset X$ is closed iff so are all intersections $A \cap X_n$, $n \in \mathbb{N}$. Not only every compact space is a k_ω space; so is every countable CW -complex, every locally compact space with countable base etc.

2.1. Theorem (Graev-Mack-Morris-Nickolas). *Let X be a k_ω -space. Then every mapping i_n is quotient, and a subset A of $F(X)$ is closed if and only if so are all intersections $A \cap F_n(X)$. In particular, $F(X)$ is a k_ω -space. \square*

The above theorem does not admit any noticeable further generalization, apart from some openly pathological cases, such as the spaces X where every G_δ set is open (the author, unpublished, 1981). In fact, it was shown in [FOST] that the mapping i_3 is not quotient even for $X = \mathbb{Q}$. Answering two question raised in this paper, the author had proved the following result [Pe7].

2.2. Theorem. *Let X be a Tychonoff space. The mapping i_2 is quotient if and only if X is a strongly collectionwise normal space (that is, every neighbourhood of the diagonal in $X \times X$ is an element of the universal uniform structure of X). \square*

However, the following property of the mappings i_n proved to be extremely useful.

2.3. Theorem (Arhangel'skiĭ [Arh2,3]). *Let Y be any subset of \tilde{X}^n such that $i_n^{-1}i_n(Y) = Y$. Then $i_n|_Y$ is a homeomorphism. \square*

A very substantial body of results concerning the structure of free topological groups over k_ω spaces have been deduced (mostly by Australian and American mathematicians) from Theorem 2.1 [Br, BrH, F, FRST2, HMo, Katz, KatzMo1,2, KatzMoN1-4, Mo10,11, Nic1,3, OrST].

The following charming theorem of Zarichnyi [Zar1,2] puts free topological groups in connection with infinite dimensional topology. The original result was stated for

free Graev topological groups, but it extends to free Markov groups immediately because topologically the group $F(X)$ is a disjoint sum of countably many copies of $F_G(X)$.

2.4. Theorem (Zarichnyĭ [Zar1]). *Let X be a compact absolute neighbourhood retract and $0 < \dim X < \infty$. Then the free topological group $F(X)$ and the free Abelian topological group $A(X)$ are homeomorphic to an open subset of the locally convex space with finest topology $\mathbb{R}^\omega = \varinjlim \mathbb{R}^n$. \square*

Returning back to general Tychonoff spaces X , one can still describe the topology of $F(X)$ with the help of mappings i_n , but in a rather non-constructive way. The following construction have been performed by Mal'cev [Mal]. Denote by \mathfrak{T}_0 the quotient topology on $F(X)$ with respect to the direct sum of the mappings i_n , $n \in \mathbb{N}$ from the space $\bigoplus_{n \in \mathbb{N}} \tilde{X}^n$. It is Hausdorff but not necessarily a group topology. Now construct recursively a transfinite chain of topologies \mathfrak{T}_λ on $F(X)$ by defining $\mathfrak{T}_{\lambda+1}$ as the quotient of the topology on $F(X) \times F(X)$ with respect to the mapping $(x, y) \mapsto x^{-1}y$, and \mathfrak{T}_τ for a limit cardinal τ as the infimum of the chain of topologies \mathfrak{T}_λ , $\lambda < \tau$. It is clear that for some λ large enough, the topology \mathfrak{T}_λ coincides with the topology of $F(X)$. Denote the least λ with this property by $\lambda(X)$. The following question is open for more than 30 years.

Problem (Mal'cev [Mal]). *Which values can $\lambda(X)$ assume?*

Seemingly, all one knows is that $\lambda(X) = 1$ for k_ω -spaces, and $\lambda(X) > 1$ for most spaces beyond this class (for instance, for $X = \mathbb{Q}$).

Another long-standing problem asked by Mal'cev in the same paper [Mal] — that of finding a constructive description of a neighbourhood system of identity of a free topological group — had been solved by Tkachenko [Tk4]. Later simpler versions of the Tkachenko's theorem have been obtained by the author [Pe7] and Sipacheva [Si1]. We will give one of the possible forms of the result. It is more reasonable to put it for free topological groups $F(X)$ over *uniform spaces* (bearing in mind that for a Tychonoff space X the free topological group over X is canonically isomorphic to the free topological group over the universal uniform space associated to X). Let $X = (X, \mathcal{U}_X)$ be a uniform space. Denote by j_2 a mapping from X^2 to $F_2(X)$ of the form $(x, y) \mapsto x^{-1}y$, and by j_2^* — a similar mapping of the form $(x, y) \mapsto xy^{-1}$. If $\Psi \in (\mathcal{U}_X)^{F(X)}$ is a family of entourages of diagonal indexed by elements of the free group over X , then we put

$$\mathcal{V}_\Psi =_{\text{def}} \cup \{x \cdot [j_2(\Psi(x)) \cup j_2^*(\Psi(x))] \cdot x^{-1} : x \in F(X)\}$$

If B_n is a sequence of subsets of some group then, following [RoeD], we denote

$$[(B_n)] =_{\text{def}} \bigcup_{n \in \mathbb{N}} \bigcup_{\pi \in S_n} B_{\pi(1)} \cdot B_{\pi(2)} \cdot \dots \cdot B_{\pi(n)},$$

where S_n is a symmetric group.

2.5. Theorem (Pestov [Pe7]). *Let (X, \mathcal{U}_X) be a uniform space. A base of neighbourhoods of identity in the free topological group $F(X)$ is formed by all sets of the form $[(\mathcal{V}_{\Psi_n})]$, where $\{\Psi_n\}$ runs over the family of all countable sequences of elements of $(\mathcal{U}_X)^{F(X)}$. \square*

2. Free subgroups. If X is a subset of a set Y , then the free group over the set of generators X is a subgroup of the free group over Y . Now let X is a topological subspace of a Tychonoff space Y ; there is still a canonical continuous group monomorphism $F(X) \hookrightarrow F(Y)$, but it need not be a topological embedding. For the first time it was noticed by Hunt and Morris [HuM], and the example was $X = (0, 1)$, $Y = [0, 1]$. Earlier Graev has shown [Gr1,2] that if Y is compact and X is closed in Y then $F(X) \hookrightarrow F(Y)$ is in isomorphic embedding of topological groups. This result was transferred to k_ω -spaces. In [Pe1,2,4] and [Nu2] it was noticed independently that a necessary condition for the monomorphism $F(X) \hookrightarrow F(Y)$ to be topological is the property that the restriction $\mathcal{U}_Y|_X$ of the universal uniformity \mathcal{U}_Y from Y to X coincides with the universal uniformity \mathcal{U}_X of X . (It is just an immediate consequence of the fact that both left and two-sided uniformities on $F(X)$ induce on X its universal uniform structure — the fact which in its turn follows from existence of Graev’s pseudometrics on $F(X)$ and was essentially known to Graev.) In the same works [Pe1,2,4] and [Nu2] it was shown, answering a question by Hardy, Morris and Thompson [HMoTh] that the above condition $\mathcal{U}_Y|_X = \mathcal{U}_X$ is sufficient in the case where X is dense in Y . A final positive answer was obtained by Uspenskii [U5] after a series of results of intermediate strength [U3].

2.6. Theorem (Uspenskii [U5]). *Let X be a topological subspace of a Tychonoff space Y . Then the monomorphism $F(X) \hookrightarrow F(Y)$ is a topological embedding if and only if $\mathcal{U}_Y|_X = \mathcal{U}_X$. \square*

A different problem has been treated by Australian and American universal arrow theorists for a long time. Let X and Y be some particular topological spaces; in which cases the free (Abelian) topological group over X can be embedded (not necessarily in a “canonical” way) as a topological subgroup into the free (Abelian) topological group over Y ? The main device under this approach was the above Theorem 2.1. We will mention just one astonishing result in this direction.

2.7. Theorem (Katz and Morris [KatzMo2]). *If X is a countable CW-complex of dimension n , then the free Abelian topological group on X is a closed subgroup of the free Abelian topological group on the closed ball B^n . \square*

3. Completeness. Our next topic can be also traced back to Graev’s papers [Gr1,2]. Graev has deduced from his description of topology of the free group over a compact space that any such free topological group is Weil complete (that is, complete with respect to the left uniform structure). The result remains true for free topological groups over k_ω -spaces.

Examples of topological groups which are complete in their two-sided uniformity but not Weil complete (and therefore admit no Weil completion at all) are known for decades, but seemingly it remains unclear whether free topological groups admit Weil completion. This question was asked by Hunt and Morris [HuM]. An obvious necessary condition for a free topological group to be Weil-complete is the Dieudonné completeness of X , that is, completeness of X w.r.t. the finest uniformity \mathcal{U}_X . The state of affairs with Weil completeness is still unclear and one has only a series of partial results stating the Weil completeness of free topological groups over particular spaces [U3].

However, it seems in a sense more natural to examine free topological groups for another form of completeness — the completeness w.r.t. the two-sided uniformity (sometimes also called *Bożejko completeness*). [Bo1]. There exists a fascinating

comprehensive result for the completeness of this kind, and the question about the validity of such a result was first asked independently by Nummela [Nu2] and the author (in oral form, talk at the Arhangel'skii's seminar on topological algebra at Moscow University, February 1981).

2.8. Theorem (Sipacheva, [Si2]). *The free topological group $F(X)$ over a Tychonoff space X is complete if and only if X is Dieudonné complete. \square*

The idea of the proof is based on the notion of a special universal arrow, $F_\rho(X)$, introduced by Tkachenko [Tk3]. Say that a subspace Y of a topological group G is *Tkachenko thin* if for every neighbourhood of identity, U , the set $\cap\{yUy^{-1} : y \in Y\}$ is a neighbourhood of identity. Consider the category of pairs (G, Y) where G is a Hausdorff topological group and Y is a Tkachenko thin subset of G , and obvious morphisms between them, and let S be the functor from this category to **Tych** of the form $(G, Y) \mapsto Y$. Now by $F_\rho(X)$ one denotes the universal arrow from a Tychonoff space X to the functor S . There is a canonical continuous algebraic isomorphism $F(X) \rightarrow F_\rho(X)$, and it can be shown without serious difficulties that the topological group $F_\rho(X)$ is complete if and only if X is Dieudonné complete [Tk3]. Sipacheva has proved that the free topological group $F(X)$ has a base of neighbourhoods of identity that are closed in the topology of the topological group $F_\rho(X)$.

Let X be a set, and let \mathcal{V} and \mathcal{W} be any two uniformities on X generating the same Tychonoff topology. (Such a triple $(X, \mathcal{V}, \mathcal{W})$ is termed sometimes a *bi-uniform space*.)

Question. *Does there exist a topological group $F(X, \mathcal{V}, \mathcal{W})$ algebraically generated by X (free over X) such that \mathcal{V} is the restriction to X of the left uniform structure of G , and \mathcal{W} is the restriction to X of the right uniform structure?*

This question can be obviously reformulated in terms of universal arrows to forgetful functors. This concept may help to understand how the completeness works.

Among other results on the algebro-topological structure of the free topological groups, let us mention a nice theorem of Tkachenko [Tk1,2] stating that the free topological group over a compact space has the c.c.c. property (together with its subsequent generalization due to Uspenskiĭ [U1]), and a characterization of such Tychonoff spaces X that the free topological group $F(X)$ embeds into a direct product of a family of separable metrizable groups [Gur].

5. Abelian case. All of the above results have, of course, their analogs for free Abelian topological groups. Moreover, one can also give a very convenient and simple description of topology of $A(X)$ which has no analog (yet?) in non-commutative case. One can define Graev metrics on $A(X)$ in the same way as for $F(X)$, and it turns out that they describe the topology of $A(X)$. It follows from this observation that the canonical morphism from $A(X)$ to the free locally convex space $L(X)$ is an embedding of $A(X)$ as a closed topological subgroup [Tk3]. Both completeness of $A(X)$ over Dieudonné complete spaces X and the Abelian analog of the subgroup theorem were established much earlier than their non-Abelian counterparts [Tk3].

The embedding $A(X) \hookrightarrow L(X)$ enables one to describe the topology of $A(X)$ as the topology of uniform convergence on all equicontinuous families of characters of $A(X)$, and this way Pontryagin-van Kampen duality comes into being. For the first

time the Pontryagin-van Kampen duality for free Abelian topological groups was studied by Nickolas [Nic2] who has shown, answering a question by Noble [No], that the topological group $A[0, 1]$ is non-reflexive (that is, does not verify the statement of Pontryagin duality theorem). Later the author had obtained the following result.

2.9. Theorem (Pestov [Pe8]). *Let X be a Dieudonné complete k -space with $\dim X = 0$. Then the free Abelian topological group $A(X)$ is reflexive.* \square

The free topological group $F(X)$ is so “regularly shaped” that one is wondering whether it satisfies any known version of noncommutative duality (Tannaka-Krein duality is known to be insufficient) or, at the very least, whether its topology can be described with the help of equicontinuous families of homomorphisms from $F(X)$ into some fixed topological group — say, $GL(\infty) = \varinjlim GL(n, \mathbb{R})$.

3. M-EQUIVALENCE AND DIMENSION

In 1945 Markov in one of his important papers [Mar2] asked whether any two Tychonoff topological spaces, X and Y , with isomorphic free topological groups $F(X)$ and $F(Y)$ are necessarily homeomorphic. Soon Graev in his no less important papers [Gr1,2] answered in the negative by constructing a whole series of pairs X, Y of spaces with $F(X) \cong F(Y)$, therefore the resulting relation of equivalence between Tychonoff spaces turned out to be substantial. Graev called such spaces X and Y *F-equivalent*; we follow the terminology due to Arhangel'skiĭ [Arh3,5,6,8] and call such spaces *Markov equivalent* or *M-equivalent*. Graev paid special attention to the pairs of spaces X, Y with Graev free topological groups isomorphic, $F_G(X) \cong F_G(Y)$; however, the distinction between the two relations of equivalence is — from the viewpoint of their topological properties — inessential. With the help of Arhangel'skiĭ's terminology, one of the central results of the Graev's paper [Gr1] can be formulated like this.

3.1. Theorem (1948, Graev). *If X and Y are M-equivalent compact metrizable spaces then $\dim X = \dim Y$.* \square

(Here $\dim X$ stands for the Lebesgue covering dimension of a space X .)

This result — as well as technique of the proof — has received a lot of attention later. The generalizations of the result came in two directions: firstly, the equivalence relation was being replaced by more and more loose ones, and secondly, the topological restrictions on the spaces X, Y were weakened.

In 1976 Joiner [Joi] noticed that the conclusion $\dim X = \dim Y$ remains true if X and Y are both *locally compact metrizable* spaces such that the free Abelian topological groups, $A(X)$ and $A(Y)$, are isomorphic. (Following Arhangel'skiĭ, we call such spaces X, Y *A-equivalent*.) Of course, *A*-equivalence of two topological spaces follows from their *M*-equivalence, because the universal arrow $A(X)$ is a composition of the universal arrow $F(X)$ and the functor of abelianization $\mathbf{TopGrp} \rightarrow \mathbf{AbTopGrp}$.

Consider the universal arrow from the free Abelian topological group $A(X)$ to the forgetful functor from the category of locally convex spaces with weak topology to $\mathbf{AbTopGrp}$. The composition of two universal arrows is obviously the free locally convex space in weak topology, $L_p(X)$. Therefore, we come to a still looser relation of equivalence between two spaces: X and Y are *l-equivalent* if $L_p(X) \cong L_p(Y)$. In 1980 Pavlovskiĭ [Pa] had shown that $\dim X = \dim Y$ if X and Y are *l-equivalent* spaces which are either *locally compact metrizable* or *separable complete metrizable*.

So far all proofs relied on a suitable refinement of the original Graev's techniques. A basically new method — that of inverse spectra — was invoked and applied to this problem by Arhangel'skiĭ [Arh3,6] who deduced from the Pavlovskiĭ's theorem the following landmark result.

3.2. Theorem (Arhangel'skiĭ 1980). *Let X and Y be l -equivalent compact spaces. Then $\dim X = \dim Y$. \square*

Independently a weaker version was obtained by Zambakhidze [Zam1]: the covering dimension of any two M -equivalent compact spaces is the same. Later this result was generalized by him to the class of Čech complete, scaly, normal, totally paracompact spaces [Zam2] (it remained not quite clear how wide this class actually was). About the same time the result had been independently somewhat generalized by Valov and Pasynkov [VP].

Further efforts have been boosted by a question asked by Arhangel'skiĭ [Arh5]: *is it true that for Tychonoff M -equivalent spaces X and Y one has $\dim X = \dim Y$?*

The answer “yes” came from the author, who proved in late 1981 [Pe3] the following result by combining and adjusting both Graev's lemma and the spectral technique of Arhangel'skiĭ:

3.3. Theorem (Pestov, 1981). *If X and Y are l -equivalent Tychonoff spaces then $\dim X = \dim Y$. \square*

As a matter of fact, the aforementioned Graev's lemma, which forms the core of the proofs, is not a single result but rather a *scheme* of results, improved and adjusted from one situation to another. We present it as it appears in [Pe5], not in the most general form possible, but in a quite elegant one.

3.4. Graev's Lemma. *If X and Y are M -equivalent Tychonoff spaces then X is a union of countably many subspaces each of which is homeomorphic to a subspace of Y . \square*

Then one is using addition theorems for covering dimension valid for spaces with countable base; to proceed from such spaces to a general situation, the Tychonoff spaces X and Y are decomposed in inverse spectra of spaces with countable base and the same dimension as $\dim X$ and $\dim Y$; the property of l -equivalence of the two limit spaces is partially delegated to the spectrum spaces, in a form strong enough to ensure a version of the Graev's lemma.

It was shown by Burov [Bu] that the result and the scheme of the proof remain true also for cohomological dimension \dim_G where the group of coefficients G is a finitely generated Abelian group (in particular, $\dim_{\mathbb{Z}} X \equiv \dim X$).

The weak dual space to $L_p(X)$ is the space of continuous functions on X with the topology of simple (pointwise) convergence, $C_p(X)$ (it follows actually from a version of the Yoneda lemma). The theory of linear topological structure of the LCS $C_p(X)$ has grown out of Banach space theory, after the following observation proved useful [Cor]: any Banach space E in weak topology is a subspace of $C_p(X)$ where X is the closed unit ball of the dual to E with weak* topology. This theory is developing now on its own, and a good survey is [Arh9]. A bridge between theory of spaces $C_p(X)$ and universal arrow theory is erected by means of the following observation: since the two LCS's in weak topology, $L_p(X)$ and $C_p(X)$, are in duality, then two topological spaces X and Y are l -equivalent if and only if $C_p(X)$ and $C_p(Y)$ are isomorphic.

Arhangel'skiĭ was first to suggest an even weaker relation of equivalence between two Tychonoff topological spaces, X and Y : two such spaces are called *u-equivalent* if the locally convex spaces $C_p(X)$ and $C_p(Y)$ are isomorphic as uniform spaces (with the natural additive uniformity). Surprisingly, it was possible to make one more step in extending the original Graev's result.

3.5. Theorem (Gul'ko, [Gu]). *If X and Y are l -equivalent Tychonoff spaces then $\dim X = \dim Y$. \square*

The proof of Gul'ko's result [Gu] develops along the same lines as the author's earlier theorem, but technically it is considerably more complicated.

One can consider even weaker relation of equivalence: two topological spaces, X and Y , are said to be *t-equivalent* [GuKh] if the locally convex spaces $C_p(X)$ and $C_p(Y)$ are *homeomorphic* as topological spaces. It is not known whether the dimension is preserved under the relation of t -equivalence. It is worth mentioning that all the aforementioned equivalence relations (those of M -, l -, u -, t -equivalence) have been distinguished from each other.

What remains still unclear, is the existence of a reasonable straightforward characterization of dimension of a Tychonoff space X in terms of the additive uniformity of the LCS $C_p(X)$, or the linear topological structure of the space $L_p(X)$, or — at the very least — the algebro-topological structure of $F(X)$. The existing proofs are in a sense obscure and do not reveal the real machinery keeping dimension preserved by the equivalence relations.

It is an opinion of the author that emerged from discussions with Gul'ko in 1991 that a complete understanding of the phenomenon of preservation of dimension is to be sought on the following way.

Conjecture. *The Lebesgue dimension of X can be expressed in terms of a certain (co)homology theory associated with the LCS in weak topology $L_p(X)$.*

It is not clear if one can use any of the already existing (co)homology theories for locally convex spaces, because a desired theory should make a sharp distinction between weak and normable topologies. For instance, the space $C(X)$ endowed with the topology of *uniform convergence on compacta* instead of the pointwise topology carries essentially no information about the dimension of X , according to celebrated Milyutin isomorphic classification theorem [Muly].

The following remarkable theorem by Pavlovskiĭ may be also suggestive; to our knowledge, no attempt has been made yet to generalize it to arbitrary CW -complexes.

3.6. Theorem (Pavlovskiĭ [Pa]). *Two polyhedra (simplicial complexes) X and Y are l -equivalent if and only if $\dim X = \dim Y$. \square*

It is well known that $\dim X \leq n$ if and only if every continuous mapping from X to a sphere S^{n+1} is homotopically trivial [vM]. The structure of the free topological groups on spheres is well understood [KatzMoN2], so the following sounds sensible.

Conjecture. *Let X be a Tychonoff topological space. Then $\dim X \leq n$ if and only if every continuous homomorphism from the free topological group $F(X)$ to the free topological group $F(S^{n+1})$ is homotopically trivial.*

Of course, similar considerations no longer work for l -equivalence because any LCS is contractible, but the above conjecture may help to reach a deeper understanding for the relation of M -equivalence.

In addition to Gel'fand-Naïmark duality, general interest to the problem of preservation of properties of topological spaces by different functors from the category **Tych** to the categories of topological algebra has been heated for a long time by the following result of Nagata [Nag].

3.7. Theorem (Nagata). *Two Tychonoff spaces X and Y are homeomorphic if and only if the topological rings $C_p(X)$ and $C_p(Y)$ are isomorphic. In other terms, the functor $C_p(\cdot)$ from **Tych** to **TopRings** is a (contravariant) inclusion functor. \square*

By considering for every Tychonoff space X the universal arrow from X to a forgetful functor from the category **TopGrp** to **Tych** sending a topological group to a topological subspace consisting of all elements of order 2, one comes to the following result [Pe12].

3.8. Theorem. *There exists a (covariant) inclusion functor $\mathbf{Tych} \rightarrow \mathbf{TopGrp}$. \square*

There is no *full* inclusion functor of such kind [Pe12].

The following question seems very natural in connection with our problematics, and it was asked independently by many (for example, by Zarichnyi in Baku-1987):

Question. *Is it true that K -groups of M -equivalent Tychonoff topological spaces are isomorphic?*

An obvious idea, to obtain the affirmative answer with the help of universal classifying groups, fails, because if G is a non-Abelian topological group and X and Y are M -equivalent, then it follows (from the Yoneda's lemma, actually) that $K(X)$ and $K(Y)$ are isomorphic as *sets*, not *groups*: the set $Hom_c(F(X), G)$ does not carry a natural groups structure because of non-commutativity of G — and the universal classifying groups in K -theory are noncommutative. (This is why a corresponding statement in [VP] is wrong.)

The general classification of topological spaces up to an M -equivalence (as well as l -equivalence and other relations mentioned in this section) seems a totally hopeless problem. For numerous results on preservation and non-preservation of particular properties of set-theoretic topology by M -equivalence, l -equivalence etc. see [Arh3,5,7,8, Gr1,2, Ok, Tch1,2]. From our point of view, there are at least two cases where such a classification may be achieved. The first is the case of l -equivalence of *CW-complexes* (in view of the Pavlovskii's theorem), and the second is the case of M -equivalence of the so-called *scattered spaces* [ArhPo] (in view of the complete classification of all countable metric spaces up to M -equivalence obtained by Graev [Gr1,2]).

4. APPLICATIONS TO GENERAL TOPOLOGICAL GROUPS

In this section we consider some applications of free topological groups to general theory of topological groups. Remark that perhaps one owes the very existence of the concept of free topological group to a stimulating applied problem of such kind: in his historical note [Mar1] Markov was openly guided by the idea of constructing the first ever example of a Hausdorff topological group whose underlying space was not normal. (The free topological group over any Tychonoff non-normal space X is such.)

Free topological groups provide flexible “building blocks” for erecting more sophisticated constructions. Also, the following theorem is of crucial importance

4.1. Theorem (Arhangel'skiĭ [Arh1]). *Let f be a quotient mapping from a topological space X onto a topological group G . Then the continuous homomorphism $\hat{f} : F(X) \rightarrow G$ extending f is open and therefore G is a topological quotient group of $F(X)$. \square*

Seemingly, analogs of this theorem exist for other types of universal arrows as well, and one is wondering whether this result can be given a universal categorical shaping. This result (and its analogs) are invaluable for examining questions of existence of couniversal objects of one or another kind.

1. NSS property. Our first example is the NSS property. A topological group G has no small subgroups if there is a neighbourhood U of the identity element e such that the only subgroup in U is $\{e\}$. This is abbreviated to *NSS*. The crucial role of the NSS property in Lie theory (especially in connection with Hilbert's Fifth Problem) is well known.

In 1971 Kaplansky wrote ([Kap], p.89): “*The following appears to be open: if G is NSS and H is a closed normal subgroup of G , is G/H NSS? This is true if in addition G is locally compact, but we shall only be able to prove it late in the game. (Of course it is an old result for Lie groups.)*”

Very soon Morris [Mo3] answered in negative by constructing a counter-example, and later he and Thompson [MoTh2] have presented the following

4.2. Theorem. *Let X be a submetrizable Tychonoff topological space (that is, a Tychonoff space admitting a continuous metric). Then the Markov free topological group $F(X)$ over X is an NSS group. \square*

It was asked in [MoTh2] whether the following result is true.

4.3. Theorem. *Each topological group is a quotient group of an NSS group. \square*

The author [Pe1,4] has deduced Theorem 4.3 from Theorems 4.2 and 4.1 (and later it turned out that such a deduction follows at once from the above Theorems 4.2 and 4.1 in conjunction with [Ju], see [Arh4,5]).

It was shown by Sipacheva and Uspenskiĭ [SiU] that both the original proof of Theorem 4.2 by Morris and Thompson [MoTh2] and the later proof proposed by Thompson [Th] are not free of certain deficiencies. In the same work [SiU] an elaborate proof of Theorem 4.2 (definitely “hard” — it relied on combinatorial technique of words in free groups) was given. Thus, both results remain valid. The concept of free Banach-Lie algebra enables us to provide a purely Lie-theoretic (and certainly “soft”) proof of Theorem 4.2 (see Section 7 below).

2. Zero-dimensionality. Our next story is about quotient groups of zero-dimensional topological groups, and it is strikingly similar to the preceding development. In 1938 Weil (see the note [Arh4] for this and the next references) claimed that open continuous homomorphisms of topological groups do not increase dimension. This statement was later refuted by Kaplan by means of a counterexample. Arhangel'skiĭ [Arh2] has shown that *every topological group with countable base is a quotient group of a zero-dimensional group*. (Zero-dimensionality here and in the sequel is understood in the sense of Lebesgue covering dimension \dim .) Possible ways to represent *any* topological group as a quotient group of a zero-dimensional one were discussed by Arhangel'skiĭ in [Arh1], but it was until late 1980 that the above conjecture remained open.

4.4. Theorem (Arhangel'skiĭ [Arh4,5]). *Any topological group is a topological quotient group of a group G with $\dim G = 0$.* \square

Subtle topological considerations involving Graev metrics on free groups played a crucial role in the proof of the main auxiliary result: *if a submetrizable topological space X is a disjoint union of a family of spaces each of which has a unique non-isolated point then $\dim F(X) = 0$* . Then the fact that every Tychonoff space is a quotient of a space with the above property is used, together with Arhangel'skiĭ's Theorem 4.1.

This result brought to life a variety of satellite theorems and examples refining the statement. Of them the most important one is, from the author's viewpoint, the following.

4.5. Theorem (Sipacheva [Si2]). *If X is a Tychonoff space and $\dim X = 0$ then $\dim F(X) = 0$.* \square

3. Topologizing a group. As the last example, we discuss a problem by Markov [Mar2] remaining open for 40 years. A subset X of a group G is called *unconditionally closed* in G if X is closed with respect to every Hausdorff group topology on G . Markov asked [Mar2] *whether a group G admits a connected group topology if and only if every unconditionally closed subgroup of G has index $\geq \mathfrak{c}$* . (Obviously, this condition is necessary.)

The first counterexample was constructed by the author in [Pe13]. Denote by $L^b(X)$ the universal arrow from a uniform space X to the forgetful functor from the category of pairs (E, Y) , E a LCS and Y a bounded subset of E (with obviously defined morphisms), to **Unif**, of the form $(E, Y) \rightarrow Y$ where Y inherits the additive uniformly from E . If G is a topological group and H a closed subgroup, then the left action of G on the quotient space G/H with a natural quotient uniform structure [RoeD] lifts to a continuous action of G on $L^b(G/H)$. The double semidirect product

$$G^\uparrow = (G \ltimes L^b(G)) \ltimes L^b(X),$$

where X is the disjoint sum of a family of copies of a quotient space of $G \ltimes L^b(G)$, serves as a counterexample to the Markov question in case where G is an uncountable totally disconnected topological group.

Later it was observed by Remus [Re] that the infinite symmetric group $S(X)$ with pointwise topology provides another — much more transparent — counterexample to the Markov's conjecture.

The author's techniques was also used by him to construct an example of a group admitting a nontrivial Hausdorff group topology but admitting no non-trivial Hausdorff metrizable topology [Pe11].

Another problem of Markov still remains open. A subset X of a topological group G is called *absolutely closed* if it is closed in the coarsest topology on G making all mappings of the form

$$x \mapsto w(x)$$

continuous as soon as $w(x)$ is a word in the alphabet formed by all elements of G and a single variable x . This topology is an analog of the Zariski topology in affine spaces; we think it is natural to call it the *Markov topology* on a group.

Problem (Markov [Mar2]). *Prove or refute the conjecture: every unconditionally closed subset of a group is absolutely closed.*

Denote by $\mathfrak{T}_M(G)$ the Markov topology on a group G , and by $\mathfrak{T}_\wedge(G)$ — the topology intersection of all Hausdorff group topologies on G . It is clear that $\mathfrak{T}_M(G) \subset \mathfrak{T}_\wedge(G)$. The Markov’s problem can be now put in other terms: *is it true that for an arbitrary group G one has $\mathfrak{T}_M(G) = \mathfrak{T}_\wedge(G)$?*

5. FREE PRODUCTS OF TOPOLOGICAL GROUPS

Graev [Gr3] presented a constructive description of the topology of the free product $G * H$ of two compact groups; also he proved a version of Kurosh subgroup theorem in the same paper. Later both results have been generalized to k_ω -groups (or, more precisely, topological groups whose underlying spaces are k_ω) [MoOTh]. It is known that those results are no longer true beyond the class of such topological groups.

One can ask about the free products of topological groups almost the same natural questions as for free topological groups: to give a reasonable description of topology in general case, to prove (or refute) that the free product of two (an arbitrary family of) complete topological groups is a complete group; to prove (or refute) that if H_α is a topological subgroup of G_α for every $\alpha \in A$ then $*_{\alpha \in A} H_\alpha$ is a topological subgroup of $*_{\alpha \in A} G_\alpha$. However, here is a question deserving, from our viewpoint, a special attention — and not only because of its respectable age.

As it is well known, the construction of free product of groups is a generalization of the construction of a free group: indeed, the free group $F(X)$ over the set X of free generators is just the free product $*_{x \in X} \mathbb{Z}_x$ of $|X|$ copies of the infinite cyclic group \mathbb{Z} . This is obviously not the case with free *topological* groups and free products of *topological* groups — unless X is discrete. In 1950 Graev mentioned this and remarked that “*the question of existence of a natural construction which would embrace both free topological groups and free products of topological groups still remains open.*” It does — for some 42 years already.

Let $\{G_x : x \in X\}$ be a family of topological groups indexed with elements of a topological space X . One would like to define the free product $*_{x \in X} G_x$ as an appropriate universal arrow in such a way that 1) in case where $G_x \cong \mathbb{Z}$ for all $x \in X$, the group $*_{x \in X} G_x$ was (naturally) topologically isomorphic to the Markov free topological group $F(X)$; 2) in case where X is a discrete topological space, $*_{x \in X} G_x$ was a usual free product of topological groups.

Our suggestion is that a clue to the above problem might be the space $\mathcal{L}(G)$ of all closed subgroups of a topological group G , endowed with an appropriate topology. This space (and, moreover, a topological lattice) has been thoroughly studied [Pr1] in connection with extending the Mal’cev Local Theorems to the case of locally compact groups. It is known that there exist numerous “natural” topologies on the set $\mathcal{L}(G)$, including the Vietoris, Chabauty, and other topologies (*loc. cit.*).

Conjecture. *The free product of a family of topological groups $\{G_x : x \in X\}$ indexed with elements of a Tychonoff topological space X can be defined as the universal arrow from $\{G_x : x \in X\}$ to the functor from the category **TopGrp** to the category of all families of topological groups indexed with elements of Tychonoff spaces (with relevant morphisms between them), which assigns to a topological group G the family $\{H : H \in \mathcal{L}(G)\}$, the space $\mathcal{L}(G)$ being endowed with an appropriate topology.*

The Graev problem can be put in connection with deformation theory and quantum groups. In quantum physics, one considers *deformations* of algebro-topological objects (such as Lie groups) as families of objects, A_{\hbar} , depending on a continuous parameter \hbar , which is assumed to be a “very small” real number approaching zero. Physically, \hbar is the Planck’s constant, and the case $\hbar = 0$ corresponds to the (quasi) classical limit of a theory; what is deformed, is the object A_0 . The absence of non-trivial deformations for classical simple Lie groups and algebras was a reason for introducing new kind of objects — the quantum groups [Dri, Man2, RTF, Ros, Wo].

While there exists a rich mathematically sound deformation theory for *Lie algebras*, deformations of Lie group are often treated at a heuristic level. The conjectural Graev construction would enable one to consider the family G_{\hbar} , $\hbar \geq 0$ of Lie groups as a veritable continuous path in the topological space $\mathcal{L}(*_{x \in X} G_x)$.

Quantum groups were introduced in mathematical physics to describe the so-called broken symmetries of physical systems. The concept of a quantum group is not something accomplished, and its development is still in progress. It is only natural, in search of more interrelations between newly explored categories of mathematical physics, to look for universal arrows between them. Does the notion of a free quantum group over a “quantum space” make sense?

7. FREE BANACH-LIE ALGEBRAS AND THEIR LIE GROUPS

The *free Banach-Lie algebra*, $\mathfrak{lie}(E)$, over a normed space E is the universal arrow from E to the forgetful functor S from the category **BLA** of complete Lie algebras endowed with submultiplicative norm to the category **Norm** of normed linear spaces.

7.1. Theorem (Pestov [Pe18]). *The free Banach-Lie algebra exists for every normed space E , and $E \hookrightarrow \mathfrak{lie}(E)$ is an isometric embedding. The Lie algebra $\mathfrak{lie}(E)$ is centerless and infinite-dimensional if $\dim E > 0$. \square*

One can also define the free Banach-Lie algebra over an arbitrary pointed metric space X (we will denote it \mathfrak{lie}_X) as the universal arrow from X to the forgetful functor from **BLA** to **Met**_{*} (zero goes to the marked point). Obviously, it is just the composition of the free Banach space and free Banach-Lie algebra arrows.

A Banach-Lie algebra \mathfrak{g} is called *enlargable* if it comes from a Banach-Lie group. Every free Banach-Lie algebra is enlargable, and we will denote the corresponding simply connected Banach-Lie group by $\mathfrak{LG}(E)$ (resp. \mathfrak{LG}_X). Since every Banach-Lie algebra \mathfrak{g} is a quotient Banach-Lie algebra of the free Banach-Lie algebra over the underlying Banach space of \mathfrak{g} , then we come to an independent proof of a result due to van Est and Świerczkowski [Ś3]: *every Banach-Lie algebra is a quotient of an enlargable Banach-Lie algebra*.

This result can be strengthened. The couniversality of the Banach space l_1 among all separable Banach spaces is well-known [LiT] (actually, it is due to the fact that l_1 is the free Banach space over a discrete metric space). Therefore, $\mathfrak{lie}(l_1)$ is a couniversal separable Banach-Lie algebra, and the universality property is transferred to the Lie group $\mathfrak{LG}(l_1)$.

7.2. Theorem. *There exists a couniversal connected separable Banach-Lie group. \square*

Of course, the same is true for groups containing a dense subset of cardinality $\leq \aleph_1$.

One can show using results of Mycielski [My] and an idea of Gel'baum [Gel] that for any metric space X , the exponential image of $X \setminus \{0\}$ in the Lie group \mathfrak{GL}_X generates an algebraically free subgroup. Now let Y be a submetrizable pointed space admitting a one-to-one continuous mapping to X . The composition of this mapping and the exponential mapping $\exp_{\mathfrak{GL}_X}$ determines a continuous monomorphism $F_G(Y) \rightarrow \mathfrak{GL}_X$, and since any Banach-Lie group has NSS property then it is shared by $F_G(Y)$. This is the promised “soft” proof of Morris-Thompson-Sipacheva-Uspenskiĭ theorem.

In view of the existence of a couniversal separable Banach-Lie group, the following question seems most natural.

Question. *Does there exist a universal separable Banach-Lie group?*

One should compare it with the following fascinating result of Uspenskiĭ [U4].

7.3. Theorem (Uspenskiĭ). *The group of isometries of the Banach space $C(I^{\aleph_0})$ endowed with the strong operator topology is a universal topological group with countable base.* \square

However, the general linear group $GL(E)$ of any Banach space E , endowed with the uniform operator topology, cannot serve as a universal Banach-Lie group because there exist enlargable separable Banach-Lie algebras \mathfrak{g} which do not admit a faithful linear representation in a Banach space [vEŠ].

The universal arrow from a Lie algebra, \mathfrak{g} , to the forgetful functor from the category of associative algebras to the category of Lie algebras is well-known; this is the universal enveloping algebra, $U(\mathfrak{g})$, of \mathfrak{g} [Dr].

It seems that little is known about a topologized version of this, that is, the universal arrow from a locally convex Lie algebra, \mathfrak{g} , to the forgetful functor from the category of locally convex associative algebras to the category of locally convex Lie algebras. Let us denote this arrow by $i_{\mathfrak{g}} : \mathfrak{g} \rightarrow U_{\mathcal{T}}(\mathfrak{g})$. Is $i_{\mathfrak{g}}$ an embedding of topological algebras? (That is, does a topological version of the Poincaré-Birkhoff-Witt theorem hold?) Is $U_{\mathcal{T}}(\mathfrak{g})$ algebraically isomorphic to $U(\mathfrak{g})$? What about the convergence of the exponential mapping for $U_{\mathcal{T}}(\mathfrak{g})$?

The only result I am aware of in this connection is the following.

7.4. Theorem [Bou]. *The universal enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional Lie algebra \mathfrak{g} can be made into a normed algebra if and only if \mathfrak{g} is nilpotent.* \square

This means that, firstly, a metric version of the universal arrow makes no sense and, secondly, in general the algebra $U_{\mathcal{T}}(\mathfrak{g})$ is non-normable even if \mathfrak{g} is finite-dimensional.

A detailed analysis of the structure of the locally convex associative algebra $U_{\mathcal{T}}(\mathfrak{g})$ would be helpful in connection with enlargability problems for \mathfrak{g} .

8. LIE-CARTAN THEOREM

The Lie-Cartan theorem says that finite-dimensional Lie algebras are enlargable, and it seems that *the question on existence of a “direct” proof of the Lie-Cartan theorem, which would be independent of both known proofs* (the cohomological one by Cartan [C] and the representation-theoretic one by Ado [Ad]), *is still open*. For a detailed discussion, see the book [Po], where it is claimed that the above question for a long time received an attention from both French and Moscow schools of Lie theorists (including Serre).

In this Section we discuss the idea of a conjectural proof based entirely on universal arrows type constructions (free topological groups and free Banach-Lie algebras).

It is well known how by means of the Hausdorff series $H(x, y)$ one can associate in the most natural and straightforward way a local Lie group (or, rather, a Lie group germ in the sense of [Ro]) to any Banach-Lie algebra \mathfrak{g} [Bou]. This is why, according to a result by Świerczkowski [Ś1], the problem of enlarging a given Banach-Lie algebra \mathfrak{g} is completely reduced to the problem of embedding a local Banach-Lie group U into a topological group G as a local topological subgroup.

Let \mathfrak{g} be a Banach-Lie algebra. Fix a neighbourhood of zero, U , such that the Hausdorff series $H(x, y)$ converges for every $x, y \in U$. (For example, set U equal to a closed ball of radius less than $(1/3)\log(3/2)$ [Bou].) Denote by $\mathcal{N}_{\mathfrak{g}}$ a closed normal subgroup generated by all elements of the form $x^{-1}[x, (-y)]y$, $x, y \in U$. Clearly, the subgroup $\mathcal{N}_{\mathfrak{g}}$ is normal in $F(\mathfrak{g})$ and does not depend on the particular choice of U . Denote by $G_{\mathfrak{g}}$ the topological group quotient of $F(\mathfrak{g})$ by $\mathcal{N}_{\mathfrak{g}}$, and by $\phi_{\mathfrak{g}} : \mathfrak{g} \rightarrow G_{\mathfrak{g}}$ the restriction of the quotient homomorphism $\pi_{\mathfrak{g}} : F(\mathfrak{g}) \rightarrow G_{\mathfrak{g}}$ to \mathfrak{g} . One can prove that $\pi_{\mathfrak{g}}$ is a universal arrow of a certain type.

It is well known (in different terms, though — [Ś2]) that the enlargability of \mathfrak{g} is equivalent to any of the following conditions: a) the intersection $\mathcal{N}_{\mathfrak{g}} \cap \mathfrak{g}$ is discrete in \mathfrak{g} ; b) the restriction of $\phi_{\mathfrak{g}}$ to a neighbourhood of zero in \mathfrak{g} is one-to-one; c) the topological group $G_{\mathfrak{g}}$ can be given a structure of an analytical Banach-Lie group in such a way that $\phi_{\mathfrak{g}}$ is a local analytical diffeomorphism; in this case $\text{Lie}(G_{\mathfrak{g}}) \cong \mathfrak{g}$, $\phi_{\mathfrak{g}} = \exp_{G_{\mathfrak{g}}}$, and $G_{\mathfrak{g}}$ is simply connected.

Although one can show that the closedness of $\mathcal{N}_{\mathfrak{g}}$ in general is not sufficient for any of these conditions to be fulfilled, it is so in the following particular case.

8.1. Theorem [Pe19]. *A Banach-Lie algebra \mathfrak{g} with finite-dimensional center is enlargable if and only if the subgroup $\mathcal{N}_{\mathfrak{g}}$ is closed in $F(\mathfrak{g})$. In this case the quotient topological group $G_{\mathfrak{g}}$ carries a natural structure of a Banach-Lie group associated to \mathfrak{g} . \square*

The proof of this result goes as follows: firstly, it is reduced to separable Banach-Lie algebras with the help of a local theorem [Pe14], and then certain perfectly direct and functorial constructions are used, including the free Banach-Lie algebra over the underlying Banach space of \mathfrak{g} , the Banach-Lie group associated to it, and their quotients.

Now only one obstacle remains between us and a direct proof of the Lie-Cartan theorem.

Conjecture. *The closedness of the subgroup $\mathcal{N}_{\mathfrak{g}}$ in the free topological group $F(\mathfrak{g})$ over the underlying topological space of a finite-dimensional Lie algebra \mathfrak{g} can be proved relying solely on the description of topology of free topological groups over finite-dimensional Euclidean spaces.*

The subgroup $\mathcal{N}_{\mathfrak{g}}$ is compactly generated; since the compact set generating it is in $F_3(\mathfrak{g})$ rather than $F_1(\mathfrak{g})$ then one should single out some additional algebro-topological property of the group $\mathcal{N}_{\mathfrak{g}}$ which would ensure the closedness (or completeness).

We already know that $\mathcal{N}_{\mathfrak{g}}$ is always closed in $F(\mathfrak{g})$ for \mathfrak{g} finite-dimensional (it follows from the Lie-Cartan theorem), and the problem looks so natural in this setting. It is so tempting to think that the genuine reason why the statement of Lie-Cartan theorem is always true for finite-dimensional Lie algebras is not (co)homological but

entirely in the realm of general topology, namely: finite dimensional Lie algebras are k_ω spaces, while infinite dimensional ones are not.

9. LOCALLY CONVEX LIE ALGEBRAS AND GROUPS

Infinite-dimensional groups play a major role in the contemporary pure and applied mathematics [Kac1,2]. Many of them cannot be given a structure of a Banach-Lie group (for example, groups of diffeomorphisms of manifolds, some of their subgroups preserving a certain differential-geometric structure, Kac-Moody groups). At the same time, in all particular examples to an infinite-dimensional group there is associated in some natural way an infinite-dimensional Lie algebra, and therefore it is appealing, to try to develop a version of Lie theory with all its attributes general enough to embrace all particular examples of infinite-dimensional groups.

Such attempts have lead to the theory of Lie groups modeled over locally convex spaces (bornological and sequentially complete [Mil]), especially over Lie groups modeled over Fréchet spaces [KoYMO]. We will call by a *Fréchet-Lie group* a group object in the category of smooth Fréchet manifolds, that is — in this case — just a smooth manifold modeled over a Fréchet space which carries a group structure such that the group operations are Fréchet C^∞ .

There is a striking difference between the Banach and Fréchet versions of Lie theory. For example, although there is a well-defined notion of the Lie algebra, $Lie(G)$, of a Fréchet-Lie group G (which is a Fréchet-Lie algebra), the exponential mapping $exp_G : Lie(G) \rightarrow G$ need not be C^∞ nor a local diffeomorphism; therefore there is in general no canonical atlas on a Fréchet-Lie group. Moreover, the following question seems to be still open:

Question [Mil, KoYMO]. *Does the exponential map $exp_G : Lie(G) \rightarrow G$ always exist for a Fréchet-Lie group G ?*

Because of such misbehaviour of Fréchet-Lie theory, some mathematicians are questioning its ability to serve as a basis for infinite-dimensional group theory. Among them is Kirillov who once (Novosibirsk, January 1988) even expressed the opinion that obtaining an answer to the above question either in positive or in negative sense would be disadvantageous all the same!

Nevertheless, we believe that this question should be answered in order to understand the proper place of Fréchet-Lie theory, and now we want to present a new, universal arrow type, construction of locally convex Lie algebras, which may give a clue.

It is convenient to present the results in the spirit of Δ -normed spaces and algebras belonging to Antonovskiĭ, Boltyanskiĭ and Sarymsakov [ABS].

Let Δ be a directed partially ordered set. A vector space E is said to be Δ -normed if there is fixed a family of seminorms $p = \{p_\delta : \delta \in \Delta\}$ with the property $p_\delta \leq p_\gamma \Leftrightarrow \delta \leq \gamma$. (The family p is called a Δ -norm because it can be treated as a single map $E \times E \rightarrow \mathbb{R}^\Delta$ where \mathbb{R}^Δ is the so-called *topological semifield*, and it satisfies close analogs of all three axioms of a usual norm.)

Let A be an algebra. We will say that a Δ -norm $p = \{p_\delta : \delta \in \Delta\}$ on A is *submultiplicative* if

- (i) for every $\delta, \gamma \in \Delta$ such that $\delta < \gamma$ and for every $x, y \in A$ one has $p_\delta(x * y) \leq p_\delta(x) + p_\delta(y)$, where $*$ denotes the binary algebra operation.

- (ii) for every $\delta \in \Delta$ there is a γ such that for every $x, y \in A$ one has $p_\delta(x * y) \leq p_\gamma(x) \cdot p_\gamma(y)$.

One can show that the topology of every locally convex topological algebra is given by an appropriate submultiplicative Δ -norm. For example, the locally multiplicatively convex topological algebras introduced by Arens and Michael [Ar, Mic] are characterized by existence of a Δ -norm with the property $p_\delta(x * y) \leq p_\delta(x) \cdot p_\delta(y)$ for all $x, y \in A$ and every $\delta \in \Delta$.

For a fixed directed set Δ the class of all complete Δ -normed Lie algebras forms a category with contracting Lie algebra homomorphisms as morphisms. We will denote this category $\Delta\mathbf{LA}$.

9.1. Theorem. *For every Δ -normed vector space (E, p) there exists a universal arrow from this space to the forgetful functor from $\Delta\mathbf{LA}$ to the category of Δ -normed spaces. It is an isometric embedding of (E, p) into a Δ -submultiplicatively normed Lie algebra $\mathfrak{lie}(E)$. \square*

In a particular case where Δ is a one-point set, the above construction coincides with the construction of a free Banach-Lie algebra over a normed space considered earlier.

If Δ has countable cofinality type (in particular, is countable) then the Lie algebra $\mathfrak{lie}(E)$ is a Fréchet-Lie algebra.

The algebra $\mathfrak{lie}(E)$ is centerless and infinite-dimensional (unless $\dim E = 1$). It is completely unclear whether such Fréchet-Lie algebras are enlargable (that is, come from Fréchet-Lie groups). The property of being centerless gives a hope that the answer is “yes,” at least in some cases. However, if $\Delta = \mathbb{N}$ and a corresponding sequence of seminorms, p , grows “fast enough,” there is a good evidence that $\mathfrak{lie}(E, p)$ can have no exponential map.

9.2. Theorem. *Let $(E, \|\cdot\|)$ be a normed space. Define a Δ -norm p , where $\Delta = \mathbb{N}$, by letting $p_n = n!\|\cdot\|$, $n \in \mathbb{N}$. Suppose there exist a Fréchet-Lie group, G , associated to the Lie algebra $\mathfrak{lie}(E)$. Then there is no exponential map $\mathfrak{lie}(E) \rightarrow G$. \square*

One can also study *free locally convex Lie algebras* over locally convex spaces, that is, universal arrows from an LCS E to the forgetful functor from the category of locally convex topological Lie algebras and continuous Lie algebra homomorphisms to the category of locally convex spaces. We will denote the free locally convex Lie algebra over E by $\mathfrak{LCLie}(E)$. If X is a Tychonoff space, then one can consider the *free locally convex Lie algebra* over X , defined either as the coomposition of the free locally convex space $L(X)$ and the free locally convex Lie algebra, or directly as the universal arrow from X to the forgetful functor from the category of locally convex topological Lie algebras and continuous Lie algebra homomorphisms to the category **Tych**. We denote this Lie algebra by \mathfrak{LCLie}_X .

P. de la Harpe has kindly drawn my attention to the following problematics.

Problem (Bourbaki [Bou]). *Is it true that every extension of a Lie algebra \mathfrak{g} by means of a \mathfrak{g} -module M is trivial (in other terms, $H^2(\mathfrak{g}, M) = (0)$ for every \mathfrak{g} -module M) if and only if \mathfrak{g} is a free Lie algebra?*

The property $H^2(\mathfrak{g}, M) = (0)$ is readily verifiable for a free Lie algebra \mathfrak{g} , but the validity of inverse implication is not known.

It is not clear yet whether free locally convex Lie algebras can help in answering the above question (supposedly in negative), but at the very least, they enjoy a similar property for *continuous* second cohomology.

9.3. Theorem. *Let X be a separable metrizable topological space, and let M be a complete normable locally convex $\mathfrak{L}\mathfrak{C}\mathfrak{L}\mathfrak{ie}_X$ -module. Then every locally convex extension of the Lie algebra $\mathfrak{L}\mathfrak{C}\mathfrak{L}\mathfrak{ie}_X$ by means of M is trivial. In particular, $H_c^2(\mathfrak{L}\mathfrak{C}\mathfrak{L}\mathfrak{ie}_X, M) = (0)$. \square*

The proof follows the argument for free Lie algebras, but the Michael Selection Theorem (Theorem 1.4.9 in [vM]) is involved.

In some cases one managed to establish the triviality of *algebraic* second cohomology for locally convex (and even Banach) Lie algebras [dlH].

10. SUPERMATHEMATICS

The (unhappy but hardly avoidable) term “supermathematics” is used to designate the mathematical background of dynamical theories with nontrivial fermionic sector in the quasi-classical limit $\hbar \rightarrow 0$. The “supermathematics” includes superalgebra, superanalysis, supergeometry etc., all of these being obtained from their “ordinary” counterparts by incorporating odd (anticommuting) quantities [BBHR, BBHRPe, B, DeW, Man1].

In one of those approaches an important role is played by the so-called ground algebras, or algebras of supernumbers; in other approach, algebras of this type come into being as algebras of superfunctions over purely odd supermanifolds. As a matter of fact, those algebras turn out to be universal arrows of a special kind, and they also find an independent application in infinite-dimensional differential geometry.

We will give necessary definitions. The term “*graded*” in this paper means “ \mathbb{Z}_2 -graded”. A graded algebra Λ is an associative algebra over the basic field \mathbb{K} together with a fixed vector space decomposition $\Lambda \cong \Lambda^0 \oplus \Lambda^1$, where Λ^0 is called the *even* and Λ^1 the *odd part (sector)* of Λ , in such a way that the *parity* \tilde{x} of any element $x \in \Lambda^0 \cup \Lambda^1$ defined by letting $x \in \Lambda^{\tilde{x}}$, $\tilde{x} \in \{0, 1\} = \mathbb{Z}_2$, meets the following restriction:

$$\tilde{x}\tilde{y} = \tilde{x} + \tilde{y}, \quad x, y \in \Lambda^0 \cup \Lambda^1$$

If in addition one has

$$xy = (-1)^{\tilde{x}\tilde{y}}yx, \quad x, y \in \Lambda^0 \cup \Lambda^1$$

then Λ is called *graded commutative*.

10.1. Theorem [Pe16,17]. *Let E be a normed space. There exists a universal arrow $\wedge_B E$ from E to the forgetful functor from the category of complete submultiplicatively normed graded commutative algebras to the category of normed spaces. It contains B as a normed subspace of the odd part $(\wedge_B E)^1$ in such a way that $E \cap \{1\}$ topologically generates $\wedge_B E$ and every linear operator f from E to the odd part Λ^1 of a complete normed associative unital graded commutative algebra Λ with a norm $\|f\|_{op} \leq 1$ extends to an even homomorphism $\hat{f}: \wedge_B E \rightarrow \Lambda$ with a norm $\|\hat{f}\| \leq 1$. \square*

Algebraically, $\wedge_B E$ is just the exterior algebra over the space E , endowed with a relevant norm and completed after that. It enjoys one more property. A *Banach-Grassmann algebra* [JP] is a complete normed associative unital graded commutative algebra Λ satisfying the following two conditions.

BG_1 (*Jadczyk-Pilch self-duality*). For any $r, s \in \mathbb{Z}_2 = \{0, 1\}$ and any bounded Λ^0 -linear operator $T: \Lambda^r \rightarrow \Lambda^s$ there exists a unique element $a \in \Lambda^{r+s}$ such that $Tx = ax$ whenever $x \in \Lambda^r$. In addition, $\|a\|$ equals the operator norm $\|T\|_{op}$ of T .

BG_2 . The algebra Λ decomposes into an l_1 type sum $\Lambda \simeq \mathbb{K} \oplus J_\Lambda^0 \oplus \Lambda^1$ where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and J_Λ^0 is the even part of the closed ideal J_Λ topologically generated by the odd part Λ^1 . In other words, for an arbitrary $x \in \Lambda$ there exist elements $x_B \in \mathbb{K}$, $x_S^0 \in J_\Lambda^0$, and $x^1 \in \Lambda^1$ such that $x = x_B + x_S^0 + x^1$ and $\|x\| = \|x_B\| + \|x_S^0\| + \|x^1\|$.

10.2. Theorem [Pe17]. *Let E be a normed space. The following conditions are equivalent:*

- (i) $\dim E = 0$;
- (ii) $\wedge_B E$ is a Banach-Grassmann algebra. \square

The algebra $\wedge_B l_1$ (denoted by B_∞) was widely used in superanalysis [JP].

The algebras of the type $\wedge_B E$ appear in infinite-dimensional differential geometry: in [KL], Klimek and Lesniewski used them for constructing Pfaffian systems over infinite-dimensional Banach spaces after it became clear that the earlier considered Pfaffians over Hilbert spaces are insufficient for applications in mathematical physics.

If one wishes to study algebras of superfunctions on purely odd (that is, including fermionic degrees of freedom only) infinite dimensional supermanifolds modeled over *locally convex* spaces, then another universal arrow comes into being. A *locally convex* graded algebra Λ carries two structures - that of a graded algebra and of locally convex space — in such a way that multiplication is continuous and both even and odd sectors are closed subspaces of Λ . A topological algebra A is called *locally multiplicatively convex*, or just *locally m-convex*, if its topology can be described by a family of all submultiplicative continuous seminorms. (Equivalently: A can be embedded into the direct product of family of normable topological algebra.) [Ar, Mic] An *Arens-Michael algebra* [He] is a complete locally m-convex algebra.

10.3. Theorem [Pe15,16]. *Let E be a locally convex space. Then there exists a universal arrow $\wedge_{AM} E$ from E to the forgetful functor from the category of graded commutative Arens-Michael algebras to the category of locally convex spaces.* \square

The two particular cases are well-known: $\wedge_{AM} \mathbb{R}^{\aleph_0}$ is the DeWitt supernumber algebra [DeW], and $\wedge_{AM} \mathbb{R}^\omega$ is the nuclear (LB) algebra considered in [KoN]. (Here \mathbb{R}^{\aleph_0} stands for the direct product of countably many copies of \mathbb{R} , and \mathbb{R}^ω denotes the direct limit $\varinjlim \mathbb{R}^n$.) In addition, in the finite-dimensional case, $\wedge_{AM} \mathbb{R}^q$ is just the Grassmann algebra with q odd generators.

Perhaps, the same sort of construction would serve as a base for study of Pfaffians on infinite dimensional locally convex spaces.

At present one of the most appealing unsolved problem in “supermathematics” is to give a unified treatment of all existing approaches to the notion of a supermanifold by viewing supermanifolds over non-trivial ground algebras Λ as superbundles over $\text{Spec } \Lambda$.

Denote by \mathcal{G} the category of finite-dimensional Grassmann algebras and unital algebra homomorphisms preserving the grading. Let $\mathbf{LCS}^{\mathcal{G}op}$ denote the category

of all contravariant functors from \mathcal{G} to the category **LCS** of locally convex spaces and continuous linear operators; we call the category **LCS** $^{\mathcal{G}^{op}}$ the category of *virtual locally convex superspaces*. Every graded locally convex space $E = E^0 \oplus E^1$ canonically becomes an object of **LCS** $^{\mathcal{G}^{op}}$, because it determines a functor of the form $\wedge(q) \mapsto [\wedge(q) \otimes E]^0$; we will identify this functor with E . The simplest non-trivial example of a virtual graded locally convex space is $\mathbb{R}^{1,1} = \mathbb{R}^1 \oplus \mathbb{R}^1$. The category **LCS** $^{\mathcal{G}^{op}}$ is a subcategory of the category **DiffLCS** $^{\mathcal{G}^{op}}$ of all contravariant functors from \mathcal{G} to the category **DiffLCS** of locally convex spaces and infinitely smooth mappings between them.

Conjecture. *The set of all morphisms in the category **DiffLCS** $^{\mathcal{G}^{op}}$ from a purely odd graded locally convex space E to $\mathbb{R}^{1,1}$ carries a natural structure of a graded locally convex algebra canonically isomorphic to the free graded commutative Arens-Michael algebra, $\wedge_{AM} E'_\beta$, on the strong dual space E'_β .*

11. C^* ALGEBRAS AND NONCOMMUTATIVE MATHEMATICS

Every normed space E admits a universal arrow to the forgetful functor from the category of (commutative) C^* -algebras and their morphisms to the category of normed spaces and contracting linear operators; we will denote it by $C^*(E)$ ($C_{com}^*(E)$, in commutative case), and refer to as *the free (commutative) C^* -algebra over a normed space*. The arrows in both cases are isomorphic embeddings. This is simply due to the two facts: firstly, every normed space E embeds into the C^* -algebra of continuous functions on the closed unit ball of the dual space E' with the weak* topology, and secondly, the class of (commutative) C^* algebras is closed under the l_∞ -type sum.

This construction is a particular case of the Blackadar's construction of a C^* -algebra defined by generators and relations [Bla]. For example, the *free C^* -algebra* over a set Γ of free generators [GM] is just the free C^* -algebra in our sense over the Banach space $l_1(\Gamma)$. In non-commutative topology [BOB] the C^* -algebras $C^*(l_1(\Gamma))$ (treated as objects of the opposite category) are viewed as noncommutative versions of Tychonoff cubes I^τ , because they are couniversal objects (universal — in the opposite category).

It is known that every free C^* -algebra is *residually finite-dimensional (RFD)*, that is, admits a family of C^* -algebra homomorphisms to finite-dimensional C^* -algebras separating points [GM]. The same is true for our more general objects.

11.1. Theorem. *For every normed space E the C^* -algebra $C^*(E)$ is residually finite-dimensional. \square*

This result seems interesting because there are few known classes of RFD C^* -algebras [ExL].

Both embeddings have been considered earlier [BIP, Ru], where the so-called matrix norms on E defined by those embeddings are denoted by MAX and MIN . This construction is especially important for the so-called quantized functional analysis [Eff], of which the idea is that all the main functional-analytic properties and results concerning Banach spaces can be expressed in terms of the universal arrow $C_{com}^*(E)$, so their non-commutative versions stated for $C^*(E)$ constitute the object of *quantized* (that is, noncommutative) functional analysis.

In this connection, it may be useful to consider two equivalence relations on Banach spaces, two such spaces E and F being equivalent iff $C^*(E) \approx C^*(F)$

(respectively, $C_{com}^*(E) \cong C_{com}^*(F)$).

If one wishes to study “quantized” theory of LCS’s then one should turn to the similar universal arrows from a given LCS E to the forgetful functor from the category of the so-called *pro- C^* -algebras* in the sense of N.C. Phillips [Ph] (just inverse limits of C^* -algebras) and their morphisms to the forgetful functor to the category of LCS’s; there are both commutative and non-commutative versions of those universal arrows.

Finally, we expect that a whole new class of examples of the so-called *quantum algebras* in the sense of Jaffe and collaborators [JLO] can be obtained by considering universal arrows from a set of data including graded normed spaces to the relevant forgetful functor.

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